J Math Chem (2013) 51:2432–2454
DOI 10.1007/s10910-013-0220-1

ORIGINAL PAPER

Two reliable wavelet methods to Fitzhugh–Nagumo (FN) and fractional FN equations

G. Hariharan · R. Rajaraman

Received: 5 April 2013 / Accepted: 21 June 2013 / Published online: 13 July 2013

Abstract Fractional reaction–diffusion equations serve as more relevant models for studying complex patterns in several fields of nonlinear sciences. In this paper, we have developed the wavelet methods to find the approximate solutions for the Fitzhugh–Nagumo (FN) and fractional FN equations. The proposed method techniques provide the solutions in rapid convergence series with computable terms. To the best of our knowledge, until now there is no rigorous wavelet solutions have been reported for the FN and fractional FN equations arising in gene propagation and model. With the help of Laplace operator and Legendre wavelets operational matrices, the FN equation is converted into an algebraic system. Finally, we have given some numerical examples to demonstrate the validity and applicability of the wavelet methods. The power of the manageable method is confirmed. Moreover, the use of the wavelet methods is found to be accurate, efficient, simple, low computation costs and computationally attractive.

Keywords Fitzhugh–Nagumo equations · Fractional FN equation · Haar wavelets · Laplace transform method · Legendre wavelets · Operational matrices · Homotopy analysis method

1 Introduction

In recent years, nonlinear reaction–diffusion equations (NLRDEs) have been widely studied and applied in science, engineering and medicine [1,2]. Reaction–diffusion

G. Hariharan (✉) · R. Rajaraman
Department of Mathematics, School of Humanities and Sciences, SASTRA University, Thanjavur 613 401, Tamilnadu, India
e-mail: hariharang2011@gmail.com; hariharan@maths.sastra.edu

R. Rajaraman
e-mail: rraja@maths.sastra.edu

equations (RDEs) are commonly applied to model the growth and spreading of biological species [3]. A fractional reaction–diffusion equation (FRDE) can be derived from a continuous-time random walk model when the transport is dispersive or a continuous-time random walk model with temporal memory and sources [4]. In recent years, the FRDE has received the applications in systems biology [5–9], chemistry, and biochemistry applications [10]. Another time-FRDE is the time-fractional Fitzhugh–Nagumo (FN) equation. It is an important NLRDE in population genetics [11], circuit theory [8,12,13], Ventricle tissue model [11,14–16] and usually used to model the transmission of nerve impulses [11,12]. In 1952, Hodgkin and Huxley [17] developed an efficient FN models for the conduction of nerve impulses along axon. They established a mathematical model to describe the membrane's behavior by considering the conduction and excitation of the fiber [18]. The FN models were derived by both Fitzhugh [12] and Nagumo et al. [19]. In recent years, these models are important nonlinear reaction–diffusion models used in circuit theory, biology and the area of population genetics [11]. The FN model equations describe the dynamical behavior near the bifurcation point for the Rayleigh-Benard convection of binary fluid mixtures [20]. But these nonlinear PDEs are difficult to get their exact solutions. So the approximation and numerical methods must be used. The numerical solutions of the NLRDEs have received considerable attention in the literature and fall into two groups: The analytical methods and the numerical ones. Analytical methods enable researchers to study the effect of different variables or parameters on the function under study easily. Recently, there are many new algorithms for NLRDEs have been proposed, for example, the Adomian decomposition Method [21,22], Pseudo-Spectral method [23,24], generalized differential transform method [25], the Homotopy Analysis method (HAM) [9,15,26,27], Haar wavelet method [21,28–33], Legendre wavelet method [34–36] and other methods [37–40]. Recently, Khan et al. [15] introduced the approximate analytical solutions of FRDEs.

Turut and Guzel [41] had applied the multivariate pade approximation (MPA) for the time-fractional RDEs. Malfliet [42] presented the solitary wave solutions of nonlinear wave equations. Elagan et al. [43] applied the generalized (G'/G) expansion method for the generalized FN equations. Hajipour and Mahmoudi [44] had used the exp-function method to FN equations.

Wavelets theory is a relatively new and as emerging area in applied mathematical research. It has been applied many different field of science and Engineering. Moreover wavelet transform establishes a connection with efficient and fast numerical algorithms.

In recent years, wavelet transforms have found their way into many different fields in science, engineering and medicine. Wavelet analysis or wavelet theory, as a relatively new and an emerging area in applied mathematical research, has received considerable attention in dealing with NLRDEs. It possesses many useful properties, such as Compact support, orthogonality, dyadic, orthonormality and multi-resolution analysis (MRA) [21,28–30]. Recently, Haar wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets [21,28–30,34–36,45] to study problems, of greater computational complexity. Among the wavelet transform families the Haar and Legendre wavelets deserve much attention. The basic idea of Legendre wavelet method is to convert the Partial differential equations to a system of algebraic equations by the operational matrices of integral or derivative [30]. The main goal is to show how wavelets and MRA can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Razzaghi and Yousefi [46,47] introduced the Legendre wavelet method for solving variational problems and constrained optimal control problems. Recently, Hariharan et al. [21,28–30] introduced the diffusion equation, convection-diffusion equation, reaction–diffusion equation, nonlinear parabolic equations, fractional Klein–Gordon equations, Sine–Gordon equations and Fisher's equation by the Haar wavelet method. Mohammadi and Hosseini [48] had showed a new Legendre wavelet operational matrix of derivative in solving singular ordinary differential equations. Parsian [49] introduced two dimensional Legendre wavelets and operational matrices of integration. Yousefi [50] introduced the Legendre wavelets for solving Lane-Emden type differential equations. Recently, Yin et al. [51] introduced the a coupled method of Laplace Transform and Legendre Wavelets for Lane–Emden type differential Equations,

In this paper, we have applied Haar and Legendre wavelets for FN and time-fractional FN equations arising in population genetics.

This paper is summarized as follows: Haar and Legendre wavelets and their properties are demonstrated in Sect. 2. Then, the methods of solution for the FN and fractional FN equations are presented in Sect. 3. In Sect. 4, the convergence analysis is described. Illustrative examples are given to demonstrate the effectiveness of the proposed method in Sect. 5. Concluding remarks are given in Sect. 6.

2 Haar and Legendre wavelets

2.1 Haar wavelet method (HWM)

Haar wavelet was a system of square wave; the first curve was marked up as $h_0(t)$, the second curve marked up as $h_1(t)$ that is

$$h_0(t) = \begin{cases} 1, & 0 \le x < 1 \\ 0, & otherwise \end{cases}$$

$$h_1(t) = \begin{cases} 1, & 0 \le x < 1/2, \\ -1, & 1/2 \le x < 1, \\ 0, & otherwise, \end{cases}$$

where $h_0(t)$ is scaling function, $h_1(t)$ is mother wavelet. In order to perform wavelet transform, Haar wavelet uses dilations and translations of function, i.e. the transform make the following function.

$$h_n(t) = h_1\left(2^j t - k\right),\ n = 2^j + k,\ j \geq 0,\ 0 \leq k < 2^j. \tag{2.1}$$

Chen and Hsiao [52] raised the ideology of operational matrix in 1975, investigated the generalized integral operational matrix, that is, the integral of matrix $\phi(t)$ can be approximated as follows:

$$\int_0^t \phi\left(t\right) dt \cong Q_\phi \phi\left(t\right) \tag{2.2}$$

where Q_ϕ is an operational matrix of one-time integral matrix $\phi(t)$, similarly, we can get operational matrix Q_ϕ^n of n-time integral of $\phi(t)$. For example, the operational matrix of $\Phi(t)$ can be expressed by following:

$$Q_\Phi = \Phi Q_B \Phi^{-1} \tag{2.3}$$

Here Q_B is the operational matrix of the block pulse function.

$$Q_{B_m} = \frac{1}{2m}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.4}$$

where m is the dimension of matrix $\Phi\left(t\right)$, and usually $m = 2^\alpha$, α is positive integer.

If $\Phi\left(t\right)$ is a unitary matrix, then $Q_\Phi = \Phi Q_B \Phi^T$, Q_Φ is a matrix with characteristic of briefness and profound utility.

For $x \in [0, 1]$, Haar wavelet function is defined as follows:

$$\begin{aligned} h_0\left(x\right) &= \frac{1}{\sqrt{m}} \\ h_i\left(x\right) &= \frac{1}{\sqrt{m}}\begin{cases} 2^{\frac{j}{2}}, & \frac{k-1}{2^j} \leq x < \frac{k-(1/2)}{2^j} \\ -2^{\frac{j}{2}}, & \frac{k-(1/2)}{2^j} \leq x < \frac{k}{2^j} \\ 0, & otherwise \end{cases} \end{aligned} \tag{2.5}$$

Integer $m = 2^j (j = 0, 1, 2 \ldots J)$ indicates the level of the wavelet; $i = 0, 1, 2, \ldots m - 1$ is the translation parameter. Maximal level of resolution is J. The index i is calculated according the formula $i = m+k-1$; in the case of minimal values $m = 1, k = 0$ we have $i = 2$, the maximal value of i is $i = 2M = 2^{J+1}$. It is assumed that the value $i = 1$ corresponds to the scaling function for which $h_1 \equiv 1 in$ [0, 1]. Let us define the collocation points $t_l = (l - 0.5)/2M$, $(l = 1, 2 \ldots 2M)$ and discretise the Haar function $h_i(x)$; in this way we get the coefficient matrix $H(i, l) = (h_i(x_l))$, which has the dimension $2M \times 2M$.

2.2 Function approximation

Any square integrable function $y(x) \in L^2[0, 1)$ can be expanded by a Haar series of infinite terms

$$y(x, t) \approx \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t), \tag{2.6}$$

where the Haar coefficients c_{ij} are determined as,

$$c_{i,j} = \int_0^1 y(x, t)\, h_i(x)\, dx \cdot \int_0^1 y(x, t)\, h_j(t)\, dt, \quad (i, j = 0, 1, 2, \ldots, m - 1) \tag{2.7}$$

are coefficients, discrete $y(x, t)$ by choosing the same step of x and t, we obtain

$$Y(x, t) = H^T(x)\, C H(t) \tag{2.8}$$

where $Y(x, t)$ is the discrete form of $y(x, t)$, and

$$H = \begin{bmatrix} h_{0,0} & h_{0,1} & \cdots & h_{0,m-1} \\ h_{1,0} & h_{1,1} & \cdots & h_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ h_{m-1,0} & h_{m-1,1} & \cdots & h_{m-1,m-1} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,m-1} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,m-1} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m-1,0} & c_{m-1,1} & \cdots & c_{m-1,m-1} \end{bmatrix}$$

C is the coefficient matrix of Y, and it can be obtained by formula:

$$C = \left(H^T\right)^{-1} Y H^{-1}. \tag{2.9}$$

H is an orthogonal matrix, then

$$C = H \cdot Y \cdot H^{-1}. \tag{2.10}$$

The operational matrix of integration P, which is a $2M$ square matrix, is defined by the equation

$$(PH)_{il} = \int_0^{t_l} h_i(t)dt$$

$$(QH)_{il} = \int_0^{t_l} dt \int_0^{t} h_i(t)dt$$

$$H_2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad P_4 = \frac{1}{16}\begin{bmatrix} 8 & -4 & -2 & -2 \\ 4 & 0 & -2 & 2 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$P_8 = \frac{1}{64}\begin{bmatrix} 32 & -16 & -8 & -8 & -4 & -4 & -4 & -4 \\ 16 & 0 & -8 & 8 & -4 & -4 & 4 & 4 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 4 & 4 & 0 & 0 & -4 & 4 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & -2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Chen and Hsiao [52] showed that the following matrix equation for calculating the matrix $P$of order m holds

$$P_{(m)} = \frac{1}{2m}\begin{pmatrix} 2mP_{(m/2)} & -H_{(m/2)} \\ H_{(m/2)}^{-1} & O \end{pmatrix}$$

where O is a null matrix of order $\frac{m}{2} \times \frac{m}{2}$,

$$H_{m\times m} \triangleq \left[h_m(t_0)h_m(t_1)\ldots h_m(t_{m-1})\right]$$

Here $\frac{i}{m} \le t < i + \frac{1}{m}$ and

$$H_{mxm}^{-1} = \frac{1}{m} H_{mxm}^T diag(r)$$

It should be noted that calculations for $P_{(m)}$ and $H_{(m)}$ must be carried out only once; after that they will be applicable for solving whatever differential equations.

2.3 Legendre wavelets

2.3.1 Legendre wavelets and its properties

The Legendre wavelets are defined by

$$\psi_{\mathrm{nm}}(\mathrm{t}) = \begin{cases} \sqrt{\mathrm{m}+\frac{1}{2}}\,2^{\frac{\mathrm{k}}{2}}\mathrm{L_m}(2^{\mathrm{k}}\mathrm{t}-\widehat{\mathrm{n}}), & \text{for } \frac{\widehat{\mathrm{n}}-1}{2^{\mathrm{k}}} \le \mathrm{t} \le \frac{\widehat{\mathrm{n}}+1}{2^{\mathrm{k}}}, \\ 0, & \text{otherwise} \end{cases} \tag{2.11}$$

where $m = 0, 1, 2, \ldots, M-1$, and $n = 1, 2, \ldots, 2^{k-1}$. The coefficient $\sqrt{m+\frac{1}{2}}$ is for orthonormality, then, the wavelets $\Psi_{k,m}(x)$ form an orthonormal basis for $\mathrm{L}^2[0,1]$. In the above formulation of Legendre wavelets, the Legendre polynomials are in the following way:

$$\begin{aligned} p_0 &= 1, \\ p_1 &= x, \\ p_{m+1}(x) &= \frac{2m+1}{m+1}xp_m(x) - \frac{m}{m+1}p_{m-1}(x). \end{aligned} \tag{2.12}$$

and $\{\mathrm{p_{m+1}(x)}\}$ are the orthogonal functions of order m, which is named the well-known shifted Legendre polynomials on the interval [0,1]. Note that, in the general form of Legendre wavelets, the dilation parameter is $\mathrm{a} = 2^{-\mathrm{j}}$ and the translation parameter is $\mathrm{b} = \mathrm{n}2^{\mathrm{j}}$ [51].

2.3.2 Two-dimensional Legendre wavelets (Yin et al. [51])

Two-dimensional Legendre wavelets in $\mathrm{L}^2(\mathrm{R})$ over the interval $[0, 1] \times [0, 1]$ as the form

$$\Psi_{n,m,n',m'}(x,y) = \begin{cases} \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)}2^{\frac{k+k'}{2}} & p_m(x)\,p_{m'}(y), \\ \frac{n-1}{2^{k-1}} \le x \le \frac{n}{2^{k-1}}, & \frac{n'-1}{2^{k-1}} \le y \le \frac{n'}{2^{k'-1}}; \\ 0, & otherwise. \end{cases} \tag{2.13}$$

and $\mathrm{m} = 0, 1, 2, \ldots, \mathrm{M}-1$, $\mathrm{m}' = 0, 1, 2, 3, \ldots \mathrm{M}'-1$, $\mathrm{n} = 1, 2, \ldots, 2^{\mathrm{k}-1}$, $\mathrm{n}' = 1, 2, \ldots 2^{\mathrm{k}'-1}$

where

$$P_m(x) = \overline{P_{m'}}\left(2^k x - 2n + 1\right), \; P_{m'}(y) = \overline{P_{m'}}\left(2^{k'} y - 2n' + 1\right), \tag{2.14}$$

$\overline{P_m}$ are Legendre functions of order m defined over the interval $[-1, 1]$.

By using two-dimensional shifted Legendre polynomials into $x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right]$ and

$y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]$, the $\int_0^1 \Psi_{n,m,n',m'}(x, y)$ can be written as

$$\int_0^1 \Psi_{n,m,n',m'}(x, y) = A_{m,m'} \cdot P_{m'}(x)\, P_{m'}(y)\, \chi_{\left[\begin{smallmatrix}\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\end{smallmatrix}\right]}(\mathrm{x, y}), \tag{2.15}$$

In which $A_{m,m'} = \sqrt{\left(m+\frac{1}{2}\right)\left(m'+\frac{1}{2}\right)2^{\frac{k+k'}{2}}}$ and $\chi_{\left[\begin{smallmatrix}\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\end{smallmatrix}\right]}(\mathrm{x, y})$ is a characteristic function defined as $\chi_{\left[\begin{smallmatrix}\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}} \\ \frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\end{smallmatrix}\right]}(\mathrm{x, y}) = \begin{cases}1, x \in \left[\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}\right], & y \in \left[\frac{n'-1}{2^{k'-1}}, \frac{n'}{2^{k'-1}}\right]; \\ \quad 0, & otherwise\end{cases}$

Two dimension Legendre Wavelets are an orthonormal set over [0, 1] × [0, 1].

$$\int_0^1\int_0^1 \Psi_{n,m,n',m'}(x, y)\,\Psi_{n_1,m_1,n_1',m_1'}(x, y)\,dxdy = \delta_{n,n_1}\delta_{n',n_1'}\delta_{m',m_1'} \tag{2.16}$$

The function u(x, y) $\in L^2(R)$ defined over [0, 1] × [0, 1] may be expanded as

$$\mathrm{u(x, y) = X(x)Y(y)} \cong \sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\sum_{n'=1}^{\infty}\sum_{m'=0}^{\infty} c_{n,m,n',m'} \quad \Psi_{n,m,n',m'}(x, y) \tag{2.17}$$

If the infinite series in Eq. (2.17) is truncated, then Eq. (2.17) can be written as

$$\mathrm{u(x, y) = X(x)Y(y)} \cong \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{n'=1}^{2^{k'-1}}\sum_{m'=0}^{M'-1} c_{n,m,n',m'} \quad \Psi_{n,m,n',m'}(x, y) \tag{2.18}$$

where $c_{n,m,n',m'} = \int_0^1\int_0^1 X(x)\, Y(y)\, \Psi_{n,m,n',m'}(x, y)\, dxdy$. The Eq. (2.18) can be expressed as the form

$$u(x, y) = c^T \cdot \Psi(x, y) \tag{2.19}$$

where C and Ψ(x,y) are coefficients matrix and wavelets vector matrix respectively. The number of dimensions of C and Ψ(x,y) are $2^{\mathrm{k}-1}2^{\mathrm{k}'-1}\mathrm{MM}' \times 1$, and given by

$$\begin{aligned} C = \big[&c_{1,0,1,0}, \ldots c_{1,0,1,M'-1}, c_{1,0,2,0}, \ldots, c_{1,0,2,M'-1}, \ldots, \\ &c_{1,0,2^{k'-1},0}, \ldots, c_{1,0,2^{k'-1},M'-1}, \ldots, c_{1,M-1,1,0}, \ldots, \\ &c_{1,M-1,1,M'-1}, c_{1,M-1,2,0}, \ldots, c_{1,M-1,2,M'-1}, \ldots, c_{1,M-1,2^{K-1},0}, \ldots, \\ &c_{1,M-1,2^{K-1},M'-1}, \ldots, c_{2,0,1,0}, \ldots, c_{2,0,1,M'-1}, c_{2,0,2,0}, \ldots, \end{aligned}$$

$$c_{2,0,2,M'-1},\ldots,c_{2,0,2^{K-1},0},\ldots,c_{2,0,2^{k-1},M'-1},\ldots,$$
$$c_{2,M-1,1,0}\ldots,c_{2,M-1,1,M'-1},c_{2,M-1,2,0},\ldots,c_{2,M-1,2,M'-1},\ldots,$$
$$c_{2,M-1,2^{k-1},0},\ldots,c_{2,M-1,2^{k-1},M'-1},\ldots,c_{2^{k-1},0,1,0},\ldots,$$
$$c_{2^{k-1},0,1M'-1},c_{2^{k-1},0,2,0},\ldots,c_{2^{k-1},0,M'-1},\ldots,$$
$$\left.c_{2^{k-1},0,2^{k-1},0},\ldots,c_{2^{k-1},M-1,2^{k'-1},M'-1}\right]^{T} \tag{2.20}$$

$$\Psi=\Big[\Psi_{1,0,1,0},\ldots,\Psi_{1,0,1,M'-1},\Psi_{1,0,2,0},\ldots\Psi_{1,0,2^{k-1},0},\ldots\Psi_{1,0,2^{k'-1},M'-1},\ldots,$$
$$\Psi_{1,M-1,1,0},\ldots\Psi_{1,M-1,1,M'-1},\Psi_{1,M-1,2,0},\ldots,\Psi_{1,M-1,2,M'-1},\ldots,$$
$$\Psi_{1,M-1,2^{k-1},0},\ldots,\Psi_{1,M-1,2^{k-1},M'-1},\ldots,\Psi_{2,0,1,0},\ldots,$$
$$\Psi_{2,0,1,M'-1},\Psi_{2,0,2,0},\ldots\Psi_{2,0,2,M'-1},\ldots,\Psi_{2,0,2^{k'-1},0},\ldots,$$
$$\Psi_{2,0,2^{k-1},M'-1},\ldots,\Psi_{2,M-1,1,0},\ldots,\Psi_{2,M-1,1,M'-1},\Psi_{2,M-1,2,0},\ldots,$$
$$\Psi_{2,M-1,2,M'-1},\ldots,\Psi_{2,M-1,2^{k'-1},0},\ldots,\Psi_{2,M-1,2^{k'-1},M'-1},$$
$$\Psi_{2^{k-1},0,1,0},\ldots,\Psi_{2^{k-1},0,1,M'-1},\Psi_{2^{k-1},0,2,0},\ldots,$$
$$\left.\Psi_{2^{k-1},0,2,M'-1},\ldots,\Psi_{2^{k-1},0,2^{k-1}},\ldots\Psi_{2^{k-1},M-1,2^{k-2},M'-1}\right]T \tag{2.21}$$

The integration of the product of two Legendre wavelet function vectors is obtained as

$$\int_0^1\int_0^1 \Psi\,(x,y)\,\Psi^T\,(x,y)\,dxdy=I \tag{2.22}$$

where I is the identity matrix.

A two-dimensional function f(x,y) defined $[0,1)\times[0,1)$ may be expanded by Legendre wavelet series as

$$f\,(x,y)=\sum_{i=1}^{2^kM}\sum_{j=1}^{2^kM}C_{ij}\,\Psi_\mathrm{i}(\mathrm{x})\Psi_\mathrm{j}(\mathrm{y})=\Psi^\mathrm{T}\,(\mathrm{x})\,\mathrm{C}\Psi(\mathrm{y}) \tag{2.23}$$

where

$$C_{ij}=\int_0^1 f\,(x,y)\,\Psi_\mathrm{i}\,(\mathrm{x})\,\mathrm{dx}\int_0^1 \mathrm{f(x,\,y)}\Psi_\mathrm{j}\,(\mathrm{y})\,\mathrm{dt} \tag{2.24}$$

Equation (2.23) can be written into the discrete form (in matrix form) by

$$f\,(x,y)=\Psi^\mathrm{T}\,(\mathrm{x})\,\mathrm{C}\Psi(\mathrm{y}) \tag{2.25}$$

where C and Ψ(t) are $2^{k-1}M \times 1$ matrices given by

$$C = \begin{bmatrix} c_{0,0} & c_{0,1} & \cdots & c_{0,2^{k-1}M} \\ c_{1,0} & c_{1,1} & \cdots & c_{1,2^{k-1}M} \\ \vdots & \vdots & \ddots & \vdots \\ c_{2^{k-1}M,0} & c_{2^{k-1}M,1} & \cdots & c_{2^{k-1}M2^{k-1}M} \end{bmatrix}$$

The two dimensional Legendre wavelet operational matrix of integration has been derived in Ref. [49].

Theorem 2.1 *Let $\Psi(x, y)$ be the two-dimensional Legendre wavelets vector defined in Eq. (2.26), we have*

$$\frac{\partial\Psi(x, y)}{\partial x} = D_x\Psi(x, y), \tag{2.26}$$

where D_x is $2^{k-1}, 2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ and has the form as follows:

$$D_x = \begin{bmatrix} D & O' & \cdots & O' \\ O' & D & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & D \end{bmatrix},$$

in which O' and D is $2^{k-1}2^{k'-1}MM' \times 2^{k-1}2^{k'-1}MM'$ matrix and the element of D is defined as follows:

$$D_{r,s} = \begin{cases} 2^k\sqrt{(2r-1)(2s-1)}I, & r = 2, 3, \ldots M; s = 1, \ldots r-1;\ r+s\, is\, odd \\ 0 & otherwise \end{cases} \tag{2.27}$$

and I, O are $2^{k'-1}M' \times 2^{k'-1}M'$ identity matrix.

Theorem 2.2 *Let Ψ (x, y) be the two-dimensional Legendre wavelets vector defined in* Eq. (2.26)*, we have*

$$\frac{\partial\Psi(x, y)}{\partial x} = D_y\Psi(\mathrm{x}, \mathrm{y}), \tag{2.28}$$

$$D_y = \begin{bmatrix} D & O' & \cdots & O' \\ O' & D & \cdots & O' \\ \vdots & \vdots & \ddots & \vdots \\ O' & O' & \cdots & D \end{bmatrix},$$

where D_y is $2^{k-1}, 2^{k'-1}$ $MM' \times 2^{k-1}2^{k'-1}MM'$ and O', D is $MM' \times MM'$ matrix is given as

$$D = \begin{bmatrix} F & O & \dots & O \\ O & F & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & F \end{bmatrix},$$

in which O and F is $M' \times M'$ matrix, and F is defined as follows:

$$F_{r,s} = \begin{cases} 2^{k'}\sqrt{(2r-1)(2s-1)}, & r = 2, \dots, M';\ S = 1, \dots, r-1;\ and\, r+s\, is\, odd \\ 0, & otherwise \end{cases} \tag{2.29}$$

Using Eqs. (2.26) and (2.28), the operational matrices for nth derivative can be derived as

$$\frac{\partial^n \Psi(x, y)}{\partial x^n} = D_x^n \Psi(x, y), \quad \frac{\partial^m \Psi(x, y)}{\partial y^m} = D_y^m \Psi(x, y)$$
$$\frac{\partial^{n+m} \Psi(x, y)}{\partial x^n \partial y^m} = D_x^n D_y^m \Psi(x, y),$$

where D^n is the nth power of matrix D.

2.4 Block pulse functions (BPFs)

The block pulse functions form a complete set of orthogonal functions which defined on the interval [0, b) by

$$b_i(t) = \begin{cases} 1, & \frac{i-1}{m}b \le t < \frac{i}{m}b, \\ 0, & elsewhere \end{cases} \tag{2.30}$$

for i = 1, 2, ..., m. It is also known that for any absolutely integrable function f(t) on [0,b) can be expanded in block pulse functions:

$$f(t) \cong \xi^T B_m(t) \tag{2.31}$$
$$\xi^T = [f_1, f_2, \dots, f_m], \quad B_m(t) = [b_1(t), b_2(t), \dots, b_m(t)] \tag{2.32}$$

where f_i are the coefficients of the block-pulse function, given by

$$f_i = \frac{m}{b} \int_0^b f(t)\, b_i(t)\, dt \tag{2.33}$$

Remark 1 Let A and B are two matrices of m × m, then $A \otimes B = \left(a_{ij} \times b_{ij}\right)_{mm}$.

Lemma 1 *Assuming f(t) and g(t) are two absolutely integrable functions, which can be expanded in block pulse function as f(t) = FB(t) and g(t) = GB(t) respectively, then we have*

$$f\,(t)\,g\,(t) = FB\,(t)\,B^T\,(t)\,G^T = HB\,(t) \tag{2.34}$$

where $H = F \otimes G$.

2.5 Approximating the nonlinear term

The Legendre wavelets can be expanded into m-set of block-pulse Functions as

$$\Psi\,(t) = \emptyset_{m\times m} B_m(t) \tag{2.35}$$

Taking the collocation points as following

$$t_i = \frac{i - 1/2}{2^{k-1}M},\; i = 1, 2, \ldots, 2^{k-1}M \tag{2.36}$$

The m-square Legendre matrix $\emptyset_{m\times m}$ is defined as

$$\emptyset_{m\times m} \cong \left[\Psi\,(t_1)\,\Psi\,(t_2)\ldots\Psi(t_{2^{k-1}M})\right] \tag{2.37}$$

The operational matrix of product of Legendre wavelets can be obtained by using the properties of BPFs, let $f\,(x,t)$ *and* $g(x,t)$ are two absolutely integrable functions, which can be expanded by Legendre wavelets as $f\,(x,t) = \Psi^T\,(x)\,F\Psi\,(t)$ and $g\,(x,t) = \Psi^T\,(x)\,G\Psi\,(t)$ respectively.

From Eq. (2.24), we have

$$f\,(x,t) = \Psi^T\,(x)\,F\Psi\,(t) = B^T\,(x)\,\emptyset^T_{m\,m}F\emptyset_{mm}B\,(t)\,, \tag{2.38}$$

$$g\,(x,t) = \Psi^T\,(x)\,G\Psi\,(t) = B^T\,(x)\,\emptyset^T_{m\,m}G\emptyset_{mm}B\,(t)\,, \tag{2.39}$$

and $F_b = \emptyset^T_{m\,m}F\emptyset_{mm}$, $G_b = \emptyset^T_{mm}G\emptyset_{mm}$, $H_b = F_b \otimes G_b$.

Then,

$$\begin{aligned} f\,(x,t)\,g\,(x,t) &= B^T H_b B\,(t)\,,\\ &= B^T\,(x)\,\emptyset^T_{m\,m} inv\left(\emptyset^T_{m\,m}\right) H_b inv\left(inv\left(\emptyset^T_{m\,m}\right) H_b inv\,(\emptyset_{m\,m})\right)\\ &\quad \times\emptyset_{m\,m}B\,(t)\\ &= \Psi^T\,(x)\,H\Psi\,(t) \end{aligned} \tag{2.40}$$

where H = $inv\left(\emptyset^T_{m\,m}\right) H_b inv\left((\emptyset_{mm})\right)$

2.6 Function approximation

A given function f(x) with the domain [0,1] can be approximated by:

$$\mathrm{f(x)}\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}c_{k,m}\Psi_{k,m}\left(x\right)=C^{T}\cdot\Psi\left(x\right). \tag{2.41}$$

If the infinite series in Eq. (2.31) is truncated, then this equation can be written as:

$$f\left(x\right)\simeq\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}c_{k,m}\Psi_{k,m}\left(x\right)=C^{T}\cdot\Psi\left(x\right). \tag{2.42}$$

where C and Ψ are the matrices of size ($2^{j-1}M \times 1$).

$$\mathrm{C}=\left[\mathrm{c}_{1,0},\mathrm{c}_{1,1},\ldots\mathrm{c}_{1,\mathrm{M}-1},\mathrm{c}_{2,0},c_{2,1},\ldots\mathrm{c}_{2,\mathrm{M}-1},\ldots\mathrm{c}_{2,1}^{\mathrm{j}-1},\ldots\mathrm{c}_{2}^{\mathrm{j}-1}{}_{,\mathrm{M}-1}\right]^{\mathrm{T}} \tag{2.43}$$

$$\Psi\left(x\right)=\left[\Psi_{1,0},\Psi_{1,1},\Psi_{2,0},\Psi_{2,1},\ldots\Psi_{2,M-1},\ldots\Psi_{2^{j-1},M-1}\right]^{\mathrm{T}}. \tag{2.44}$$

2.6.1 Definitions of fractional derivatives and integrals

In this paper, we shall use the Caputo derivative D^{α} proposed by Caputo in his work on the theory of viscoelasticity. In the development of theories of fractional derivatives and integrals, it appears many definitions such as Riemann-Liouville and Caputo fractional differential-integral definition as follows.

(1) Riemann–Liouville definition:

$${}_{a}^{R}D_{t}^{\alpha}f\left(t\right)=\begin{cases}\frac{d^{m}f(t)}{dt^{m}}, & \alpha=m\in N;\\ \frac{d^{m}}{dt^{m}}\frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}\frac{f(T)}{(t-T)^{\alpha-m+1}}dT, & 0\leq m-1<\alpha<m.\end{cases}$$

Fractional integral of order α is as follows:

$${}_{a}^{R}I_{t}^{\alpha}f\left(t\right)=\frac{1}{\Gamma\left(-\alpha\right)}\int_{0}^{t}\left(t-T\right)^{-\alpha-1}f\left(T\right)dT,\quad\alpha<0.$$

(2) Caputo definition:

$${}_{a}^{c}D_{t}^{\alpha}f\left(t\right)=\begin{cases}\frac{d^{m}f(t)}{dt^{m}}, & \alpha=m\in N;\\ \frac{1}{\Gamma(m-\alpha)}\int_{a}^{t}\frac{f^{(m)}(T)}{(t-T)^{\alpha-m+1}}dT, & 0\leq m-1<\alpha<m.\end{cases}$$

3 Method of solution

3.1 Solving Fitzhugh–Nagumo (FN) equation by the Haar wavelet method (HWM)

We consider the FN equation [31]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\,(u-\alpha)\,(1-u) \tag{3.1}$$

Since $u\,(x,t) \in L^2\,(R)$, we suppose

$$u(x,t) \approx \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} c_{ij} h_i(x) h_j(t), \tag{3.2}$$

Then we can obtain the discrete form of Eq. (3.13) by taking step $\Delta = 1/m$ of x, t, there is

$$u = H^T\,(x)\,CH\,(t) \tag{3.3}$$

$$\begin{aligned} \frac{\partial u}{\partial t} &\approx H^T\,(x)\,C\frac{\partial}{\partial t}H\,(t) \\ &= H^T\,(x)\,CQ_H^{-1}H\,(t) \end{aligned} \tag{3.4}$$

$$\frac{\partial^2 u}{\partial x^2} \approx H^T\,(x)\left(Q_H^{-2}\right)^T CH\,(t) \tag{3.5}$$

Substitute Eqs. (3.3)–(3.5) into Eq. (3.1), there is

$$\begin{aligned} &H^T\,(x)\,CQ_H^{-1}H\,(t) \\ &\quad = H^T\,(x)\left(Q_H^{-2}\right)^T CH\,(t) + H^T\,(x)\,CH\,(t)\left(H^T\,(x)\,CH\,(t)-\alpha\right) \\ &\qquad \times\left(1-H^T\,(x)\,CH\,(t)\right) \end{aligned} \tag{3.6}$$

From the above formula, the wavelet coefficient C can be calculated successively.

3.2 Solving Fitzhugh–Nagumo (FN) equation by the LLWM

We consider the Eq. (3.1) [31]

Taking Laplace transform on both sides of Eq. (3.1), we get

$$sL(u) - u(x,0) = L[u_{xx} - u^2 - u^3 - u\alpha + u^2\alpha] \tag{3.7}$$

$$sL(u) = u(x,0) + L[u_{xx} - u^2 - u^3 - u\alpha + u^2\alpha] \tag{3.8}$$

$$L(u) = s^{-1}\,(u(x,0)) + s^{-1}\left(L[u_{xx} - u^2 - u^3 - u\alpha + u^2\alpha]\right) \tag{3.9}$$

Taking inverse Laplace transform on both sides of Eq. (3.9)

$$u = u(x,0) + L^{-1}\left(s^{-1}L\left[u_{xx} - u^2 - u^3 - u\alpha + u^2\alpha\right]\right) \tag{3.10}$$

Because

$$\begin{aligned} L^{-1}\left[s^{-1}(t^n)\right] &= L^{-1}\left(n!s^{-(n+2)}\right) \\ &= \frac{1}{n+1}t^{n+1}; \quad (n = 0, 1, 2, \ldots) \end{aligned} \tag{3.11}$$

We have

$$L^{-1}[s^{-1}L(\cdot)] = \int_0^t (\cdot)dt \tag{3.12}$$

From Eq. (3.10), we gain

$$u = u(x,0) + L^{-1}(s^{-1}L[u_{xx} + g(u)]), \tag{3.13}$$

where $g(u) = u^2\alpha - u\alpha + u^2 - u^3$
Using the Legendre wavelets method,

$$u = C^T\psi(x,t) \tag{3.14}$$

$$\left.\begin{aligned} u(x,0) &= S^T\psi(x,t) \\ g(u) &= G^T\psi(x,t) \end{aligned}\right\} \tag{3.15}$$

Substituting Eqs. (3.14) and (3.15) in Eq. (3.13), we obtain

$$C^T = S^T + (C^T Dx^2 + G^T)P_t^2 \tag{3.16}$$

Here G^T has a nonlinear relation with C. When we solve a nonlinear algebraic system, we get the solution is more complex and large computation time. In order to overcome the above drawbacks, we introduce an approximation formula as follows:

$$u_{n+1} = u(x,0) + \Pi\left(\frac{\partial^2 u_n}{\partial x^2} + g(u_n)\right), \tag{3.17}$$

where $g(u) = u^2\alpha - u\alpha + u^2 - u^3$

We start the first iteration; an initial guess of the solution of u_0 is required. We select

$u_0 = u(x,0)$, and expanding u by Legendre wavelets, we gain

$$C_{n+1}^T = C_0^T + \left[C_n^T D_x^2 + G_n^T\right] P_t^2 \tag{3.18}$$

4 Convergence analysis

$$u^* = u_0 + \Pi\left(u^*_{xx} + g\left(u^*\right)\right) \tag{4.1}$$

and

$$u_{n+1} = u_0 + \Pi\left((u_n)_{xx} + g(u_n)\right) \tag{4.2}$$

Subtracting Eq. (4.1) from Eq.(4.2), we obtain

$$u_{n+1} - u^* = \Pi\left[\left(u_n - u^*\right)_{xx} + \left(g(u_n) - g\left(u^*\right)\right)\right] \tag{4.3}$$

Using Lispschitz condition,

$$\left\|g(u_n) - g(u^*)\right\| \le \gamma \left\|u_n - u^*\right\| \tag{4.4}$$

We have

$$\left\|u_{n+1} - u^*\right\| \le \left\|\prod (u_n - u^*)_{xx}\right\| + \left\|\prod (g(u_n) - g(u^*)\right\| \tag{4.5}$$

$$\le \left\|\prod (u_n - u^*)_{xx}\right\| + \gamma \left\|u_n - u^*\right\| \tag{4.6}$$

Let

$$u_{n+1} = C^T_{n+1}\psi(x,t) \tag{4.7}$$

$$u^* = C^T\psi(x,t) \tag{4.8}$$

$$\in^T_{n+1} = C^T_{n+1} - C^T \tag{4.9}$$

From Eq. (4.6), we obtain the formula

$$\in^T_{n+1} \le \in^T_n \left\|D^2_x P^2_t + \gamma P^2_t\right\| \tag{4.10}$$

By recursion, we get

$$\in^T_{n+1} \le \in^T_n \left\|D^2_x P^2_t + \gamma P^2_t\right\|^n \in_0 \tag{4.11}$$

When $\underset{n\to\infty}{Lim} \left\|D^2_x P^2_t + \gamma P^2_t\right\|^n = 0$, the series solution of Eq. (3.1) using the LLWM converges to $u^*(x)$. By using the definitions of D_x and P_t, we can get the value of γ.

Suppose $k = k' = 1$ and $M = M'$, the maximum element of D_x and P_t is $2\sqrt{(2M-1)(2M-3)}$ and 0.5 respectively.

5 Numerical examples

In this section, three examples are given for demonstrating the validity and applicability of the proposed wavelet methods.

Example 5.1 Consider the FN equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u-\alpha)(1-u), \quad 0 < \alpha < 1 \tag{5.1}$$

Subject to the initial condition

$$u(x,0) = f(x) \tag{5.2}$$

Using HAM, the exact solution in a closed form is given by

$$u(x,t) = \frac{1}{1 + e^{\left(\frac{-x+ct}{\sqrt{2}}\right)}}, \tag{5.3}$$

which is full agreement with the results in [22], where $c = \sqrt{2}\left(\frac{1}{2} - \alpha\right)$.

The Haar wavelet scheme is given by

$$\begin{aligned} &H^T(x)\, C Q_H^{-1} H(t) \\ &\quad = H^T(x)\left(Q_H^{-2}\right)^T C H(t) + H^T(x)\, C H(t)\left(H^T(x)\, C H(t) - \alpha\right) \\ &\quad\quad \times \left(1 - H^T(x)\, C H(t)\right) \end{aligned}$$

Our proposed wavelet methods HWM and LLWM can be compared with Wazwaz and Gorguis results (see Ref. [22]) and Mehmet Merdan results (see Ref. [53])

Example 5.2 We consider the time-fractional FN equation (Wazwaz and Gorguis [22])

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + u(u-\mu)(1-u), \ \mu > 0, \ 0 < \alpha \le 1, \ t > 0, \ x \in \Re \tag{5.4}$$

Subject to the initial condition

$$u(x,0) = \frac{1}{1 + e^{\left(\frac{-x}{\sqrt{2}}\right)}} \tag{5.5}$$

As $\alpha \to 1$ and $h = -1$, the exact solution of Eq. (5.4) in a closed form by the HAM is given by

Table 1 Numerical values when $\alpha = 0.5$ and $\mu = 0.45$ for Example 5.2

x	t	Exact solution u_{HAM}	Numerical $u_{HWM}(m = 32)$	Numerical $u_{LLWM}(k = 1$ and $M = 3)$
0.001	0.001	0.50026800	0.50026801	0.50026802
0.002	0.002	0.50027473	0.50027473	0.50027471
0.003	0.003	0.50017995	0.50017993	0.50017993
0.004	0.004	0.50010722	0.50010721	0.50010722
0.005	0.005	0.5002438	0.5002435	0.5002435
0.006	0.006	0.49993390	0.49993390	0.49993391
0.007	0.007	0.49998373	0.49998374	0.49998372
0.008	0.008	0.49992356	0.49992357	0.49992353
0.009	0.009	0.49973585	0.49973588	0.49973584
0.01	0.01	0.49963020	0.49963021	0.49963022

Table 2 Numerical values when $\alpha = 0.75$ and $\mu = 0.45$ for Example 5.2

x	t	Exact solution u_{HAM}	Numerical $u_{HWM}(m = 32)$	Numerical $u_{LLWM}(k = 1$ and $M = 3)$
0.001	0.001	0.49989967	0.49989969	0.49989966
0.002	0.002	0.49977499	0.49977498	0.49977497
0.003	0.003	0.49964385	0.49964388	0.49964388
0.004	0.004	0.49950898	0.49950889	0.49950892
0.005	0.005	0.49937151	0.49937150	0.49937151
0.006	0.006	0.49923211	0.49923210	0.49923210
0.007	0.007	0.49909115	0.49909111	0.49909113
0.008	0.008	0.49894892	0.49894890	0.49894891
0.009	0.009	0.49880561	0.49880558	0.49880560
0.01	0.01	0.49866137	0.49866132	0.49866132

$$u(x, t) = \frac{1}{1 + e^{\left(\frac{x}{\sqrt{2}} + \gamma t\right)}}, \tag{5.6}$$

where $\gamma = \frac{1}{\sqrt{2}} - \sqrt{2}\mu$.

Our proposed methods HWM and LLWM can be compared with Wazwaz and Gorguis results (see Ref. [22]), Soliman's results [25] and Mehmet Merdan results (see Ref. [53]).

The numerical solutions of time-fractional FN (Example 5.2) for different values of α, (That is,$\alpha = 0.5$, $\alpha = 0.75$ and $\alpha = 1.0$) and different values of t with $\mu = 0.45$ are presented in Tables 1, 2 and 3. Table 4 shows the numerical solutions of time-fractional FN equation for $\alpha = 0.7$, $t = 0.2$ and $\mu = 0.6$.The wavelet methods like

Table 3 Numerical values when $\alpha = 1.0$ and $\mu = 0.45$ for Example 5.2

x	t	Exact solution u_{HAM}	Numerical $u_{HWM}(m = 32)$	Numerical $u_{LLWM}(k = 1$ and$M = 3)$
0.001	0.001	0.49983572	0.49983568	0.49983571
0.002	0.002	0.49967144	0.49967142	0.49967145
0.003	0.003	0.49950716	0.49950715	0.49950715
0.004	0.004	0.49934288	0.49934286	0.49934288
0.005	0.005	0.49917859	0.49917857	0.49917858
0.006	0.006	0.49901431	0.49901431	0.49901430
0.007	0.007	0.49988501	0.49988501	0.49988502
0.008	0.008	0.49868574	0.49868570	0.49868571
0.009	0.009	0.49852145	0.49852142	0.49852146
0.01	0.01	0.49835716	0.49835710	0.49835712

Table 4 Numerical values when $\alpha = 0.7,\ t = 0.2$ and $\mu = 0.6$ for Example 5.2

x	Exact solution u_{HAM}	Numerical $u_{HWM}(m = 32)$	Numerical $u_{LLWM}(k = 1$ and$M = 3)$
0.00	0.49065905	0.49065902	0.49065903
0.25	0.44663406	0.44663402	0.44663406
0.50	0.40245145	0.40245144	0.40245144
0.75	0.36175911	0.36175910	0.36175912
1.0	0.32225612	0.32225611	0.32225610

HWM and LLWM results are in excellent agreement with the exact solution and those obtained by the HAM.

All the numerical experiments presented in this section were computed in double precision with some MATLAB codes on a personal computer System Vostro 1,400 Processor $86\times$ Family 6 Model 15 Stepping 13 Genuine Intel $\sim 1,596$ MHz.

6 Conclusion

Two reliable wavelet methods have been successfully employed to obtain the numerical solutions of FN and time-fractional FN equations arising in population dynamics. The proposed schemes are the capability to overcome the difficulty arising in calculating the integral values while dealing with nonlinear problems. These two wavelet methods show higher efficiency than the traditional Legendre wavelet method for solving nonlinear PDEs. Numerical example illustrates the powerful of the proposed schemes. Also this paper illustrates the validity and excellent potential of the wavelet methods for nonlinear and fractional PDEs. The numerical solutions obtained using the proposed method show that the solutions are in very good coincidence with the exact solution. In addition the calculations involved in HWM and LLWM are simple, straight forward and low computation cost. In Sect. 4, we have developed the convergence of the proposed algorithm.

7 Appendix

7.1 Basic idea of Homotopy analysis method (HAM)

In this section the basic ideas of the HAM are presented. Here a description of the method is given to handle the general nonlinear problem.

$$Nu_0\left(t\right)=0,\ \ t>0 \tag{7.1}$$

where N is a nonlinear operator and $u_0(t)$ is unknown function of the independent variable t.

7.2 Zero-order deformation equation

Let $u_0(t)$ denote the initial guess of the exact solution of Eq. (7.1), h $\neq 0$ an auxiliary parameter, $H(t)\neq 0$ an auxiliary function and L is an auxiliary linear operator with the property.

$$L\left(f\left(t\right)\right)=0,\quad f\left(t\right)=0. \tag{7.2}$$

The auxiliary parameter h, the auxiliary function $H(t)$, and the auxiliary linear operator L play an important role within the HAM to adjust and control the convergence region of solution series. Liao [26] constructs, using $q\in[0,1]$ as an embedding parameter, the so-called zero-order deformation equation.

$$(1-q)\,L[(\emptyset\left(t;q\right)-u_0\left(t\right)]=qhH\left(t\right)N[(\emptyset\left(t;q\right)]\,, \tag{7.3}$$

where $\emptyset\left(t;q\right)$ is the solution which depends on h, $H\left(t\right)$, L, $u_0(t)$ and q. When q=0, the zero-order deformation Eq. (7.2) becomes

$$\emptyset\left(t;0\right)=u_0(t), \tag{7.4}$$

and when q=1, since h $\neq 0$ and $H(t)\neq 0$, the zero-order deformation Eq. (7.1) reduces to,

$$N\left[\emptyset\left(t;1\right)\right]=0, \tag{7.5}$$

So, $\emptyset\left(t;1\right)$ is exactly the solution of the nonlinear equation. Define the so-called mth order deformation derivatives.

$$u_m\left(t\right)=\frac{1}{m!}\frac{\partial^m\emptyset\left(t;q\right)}{\partial q^m} \tag{7.6}$$

If the power series Eq. (7.3) of $\emptyset\,(t;q)$ converges at q = 1, then we gets the following series solution:

$$u\,(t) = u_0\,(t) + \sum_{m=1}^{\infty} u_m\,(t)\,. \tag{7.7}$$

where the terms $u_m\,(t)$ can be determined by the so-called high order deformation described below.

7.3 High-order deformation equation

Define the vector,

$$\overrightarrow{u_n} = \{u_0\,(t)\,, u_1\,(t)\,, u_2\,(t) \ldots u_n\,(t) \tag{7.8}$$

Differentiating Eq. (7.3) m times with respect to embedding parameter q, the setting q = 0 and dividing them by $m!$, we have the so-called mth order deformation equation.

$$L\left[u_m\,(t) - \aleph_m u_{m-1}\,(t)\right] = hH\,(t)\,R_m\left(\overrightarrow{u_m}, t\right), \tag{7.9}$$

where

$$\aleph_m = \begin{cases} o, & m \le 1 \\ 1, & otherwise \end{cases} \tag{7.10}$$

and

$$R_m\left(\overrightarrow{u_m}, t\right) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\emptyset(t;q)]}{\partial q^{m-1}} \tag{7.11}$$

For any given nonlinear operator$N$, the term $R_m\left(\overrightarrow{u_m}, t\right)$ can be easily expressed by Eq. (7.11). Thus, we can gain $u_1\,(t)\,, u_2\,(t) \ldots\ldots$ by means of solving the linear high-order deformation with one after the other order in order. The m^{th} –order approximation of u (t) is given by

$$u\,(t) = \sum_{k=0}^{m} u_k\,(t) \tag{7.12}$$

ADM, VIM and HPM are special cases of HAM when we set $h = -1$ and $H\,(r,t) = 1$. We will get the same solutions for all the problems by above methods when we set h $= -1$ and $H\,(r,t) = 1$. When the base functions are introduced the $H\,(r,t) = 1$ is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

References

1. S.Z. Rida, A.M.A. El-Sayed, A.A.M. Arafa, On the solutions of time-fractional reaction-diffusion equations. Commun. Nonlinear Sci. Numer. Simul. **5**(12), 3847–3854 (2010)
2. A. Cuyt, L. Wuytack, *Nonlinear Methods in Numerical Analysis* (Elsevier, Amsterdam, 1987)
3. J.D. Murray, *Lectures on Non-linear Differential Equation Models in Biology* (Clarenden, Oxford, 1977)
4. B.I. Henry, S.L. Wearne, Fractional reaction-diffusion. Physica A **276**(3–4), 448–455 (2000)
5. F.C. Meral, T.J. Royston, R. Magin, Fractional calculus in viscoelasticity: an experimental study. Commun. Nonlinear Sci. Numer. Simul. **15**(4), 939–945 (2010)
6. B. Baeumer, M. Kovács, M.M. Meerschaert, Numerical solutions for fractional reaction-diffusion equations. Comput. Math. Appl. **55**(10), 2212–2226 (2008)
7. M.D. Bramson, Maximal displacement of branching Brownian motion. Commun. Pure Appl. Math. **31**(5), 531–581 (1978)
8. S. Momani, R. Qaralleh, Numerical approximations and Padé approximants for a fractional population growth model. Appl. Math. Model. **31**(9), 1907–1914 (2007)
9. G. Hariharan, The homotopy analysis method applied to the Kolmogorov–Petrovskii–Piskunov (KPP) and fractional KPP equations. J. Math. Chem. **51**, 992–1000 (2013). doi:10.1007/s10910-012-0132-5
10. S.B. Yuste, L. Acedo, K. Lindenberg, Reaction front in an A+B → C reaction-subdiffusion process. Phys. Rev. E. **69**(3), part 2, Article ID 036126 (2004)
11. D.J. Aronson, H.F. Weinberg, *Nonlinear Diffusion in Population Genetics Combustion and Never Pulse Propagation* (Springer, New York, 1988)
12. R. Fitzhugh, Impulse and physiological states in theoretical models of nerve membrane. Biophys. J. **1**(6), 445–466 (1961)
13. R. Fitzhugh, *Biological Engineering* (McGraw-Hill, New York, 1969), pp. 1–85
14. K. Seki, M. Wojcik, M. Tachiya, Fractional reaction-diffusion equation. J. Chem. Phys. **119**, 2165–2174 (2003)
15. N.A. Khan, N.-U. Khan, A. Ara, M. Jamil, Approximate analytical solutions of fractional reaction-diffusion equations. J. King Saud Univ. Sci. **24**, 111–118 (2012)
16. A. Slavova, P. Zecca, CNN model for studying dynamics and traveling wave solutions of FitzHugh–Nagumo equation. J. Comput. Appl. Math. **151**, 13–24 (2003)
17. A.L. Hodgkin, A.F. Huxley, Aquantitive description of membrane current and its application th conduction and excitation in nerve. J. Physiol. **117**, 500 (1952)
18. A. Panfilov, P. Hogeweg, Spiral breakup in a modified Fitzhugh–Nagumo model. Phys. Lett. A **176**, 295–299 (1993)
19. J.S. Nagumo, S. Arimoto, S. Yoshizawa, An active pulse transmission line simulating nerve axon. Proc. IRE **50**, 2061–2071 (1962)
20. H.C. Rosu, O. Cornejo-Perez, Super symmetric pairing of kinks for polynomial nonlinearities. Phys. Rev. E **71**, 1–13 (2005)
21. G. Hariharan, K. Kannan, Haar wavelet method for solving Fisher's equation. Appl. Math. Comput. **211**, 284–292 (2009)
22. A.M. Wazwaz, A. Gorguis, An analytical study of Fisher's equation by using Adomian decomposition method. Appl. Math. Comput. **154**, 609–620 (2004)
23. D. Olmos, B. Shizgal, Pseudospectral method of solution of the Fitzhugh–Nagumo equation. Math. Comput. Simul. **79**, 2258–2278 (2009)
24. D. Olmos, B. Shizgal, A spectral method of solution of Fisher's equation. J. Comput. Appl. Math. **193**, 219–242 (2006)
25. A.A. Soliman, Numerical simulation of the FitzHugh–Nagumo Equations. Abstr. Appl. Anal. Article ID 762516, (2012) 13 p. doi:10.1155/2012/762516
26. S.J. Liao, *Beyond Perturbation: Introduction to Homotopy Analysis Method* (CRC Press/Chapman and Hall, Boca Raton, 2004)
27. S. Abbasbandy, Soliton solutions for the Fitzhugh–Nagumo equation with the homotopy analysis method. Appl. Math. Model. **32**, 2706–2714 (2008)
28. G. Hariharan, K. Kannan, K. Sharma, Haar wavelet in estimating the depth profile of soil temperature. Appl. Math. Comput. **210**, 119–225 (2009)
29. G. Hariharan, K. Kannan, Haar wavelet method for solving nonlinear parabolic equations. J. Math. Chem. **48**, 1044–1061 (2010)

30. G. Hariharan, K. Kannan, A comparative study of a Haar Wavelet Method and a Restrictive Taylor's Series Method for solving Convection-diffusion Equations. Int. J. Comput. Methods Eng. Sci. Mech. **11**(4), 173–184 (2010)
31. G. Hariharan, K. Kannan, Haar wavelet method for solving FitzHugh–Nagumo Equation. World Acad. Sci. Eng. Technol. **43**, 560–564 (2010)
32. U. Lepik, Numerical solution of evolution equations by the Haar wavelet method. J. Appl. Math. Comput. **185**, 695–704 (2007)
33. U. Lepik, Solving PDEs with the aid of two-dimensional Haar wavelets. Comput. Math. Appl. **61**, 1873–1879 (2011)
34. H. Jafari, M. Soleymanivaraki, M.A. Firoozjaee, Legendre wavelets for solving fractional differential equations. J. Appl. Math. **7**(4), 65–70 (2011, Winter)
35. Y. Yang, Solving a nonlinear multi-order fractional differential equation using Legendre Pseudo-Spectral method. Appl. Math. **4**, 113–118 (2013). doi:10.4236/am.2013.41020
36. M.H. Heydari, M.R. Hooshmandasl, F.M. Maalek Ghaini, F. Mohammadi, Wavelet collocation method for solving multiorder fractional differential equations. J. Appl. Math. Article ID 54240, (2012). doi:10.1155/2012/542401
37. H.A. Abdusalam, Analytic and approximate solutions for Nagumo telegraph reaction diffusion equation. Appl. Math. Comput. **157**, 515–522 (2004)
38. D.Y. Chen, Y. Gu, Cole-Hopf quotient and exact solutions of the generalized Fitzhugh-Nagumo equations. Acta Math. Sci. **19**(1), 7–14 (1999)
39. H. Li, Y. Guo, New exact solutions to the Fitzhugh–Nagumo equation. Appl. Math. Comput. **180**, 524–528 (2006)
40. M. Shih, E. Momoniat, F.M. Mahomed, Approximate conditional symmetries and approximate solutions of the perturbed Fitzhugh–Nagumo equation. J. Math. Phys. **46**, 023503 (2005)
41. V. Turut, N. Guzel, Comparing numerical methods for solving time-fractional reaction-diffusion equations. ISRN Mathematical Analysis (2012) Article ID 737206. doi:10.5402/2012/737206
42. W. Malfliet, Solitary wave solutions of nonlinear wave equations. Am. J. Phys. **60**(7), 650–654 (1992)
43. S.K. Elagan, M. Sayed, Y.S. Hamed, An innovative solutions for the generalized FitzHugh-Nagumo Equation by using the Generalized $\left(G'/G\right)$-Expansion Method. Appl. Math. **2**, 470–474 (2011)
44. A. Hajipour, S.M. Mahmoudi, Application of Exp-function method to Fitzhugh–Nagumo equation. World Appl. Sci. J. **9**(1), 113–117 (2010)
45. K. Maleknejad, S. Sohrabi, Numerical solution of Fredholm integral equations of the first kind by using Legendre wavelets. Appl. Math. Comput. **186**, 836–843 (2007)
46. M. Razzaghi, S. Yousefi, The Legendre wavelets operational matrix of integration. Int. J. Syst. Sci. **32**, 495–502 (2001)
47. M. Razzaghi, S. Yousefi, The Legendre wavelets direct method for variational problems. Math. Comput. Simul. **53**, 185–192 (2000)
48. F. Mohammadi, M.M. Hosseini, A new Legendre wavelet operational matrix of derivative and its applications in solving the singular ordinary differential equations. J. Frankl. Inst. **348**, 1787–1796 (2011)
49. H. Parsian, Two dimension Legendre wavelets and operational matrices of integration. Acta Mathematica Academiae Paedagogicae Nyíregyháziensis **21**, 101–106 (2005)
50. S.A. Yousefi, Legendre wavelets method for solving differential equations of Lane–Emden type. App. Math. Comput. **181**, 1417–1442 (2006)
51. F. Yin, J. Song, F. Lu, H. Leng, A coupled method of Laplace transform and legendre wavelets for Lane-Emden-Type differential equations. J. Appl. Math. Article ID 163821 (2012). doi:10.1155/2012/163821
52. C.F. Chen, C.H. Hsiao, Haar wavelet method for solving lumped and distributed-parameter systems. IEEE Proc. Part D **144**(1), 87–94 (1997)
53. M. Merdan, Solutions of time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative. Int. J. Phys. Sci. **7**(15), 2317–2326 (2012)